Metrization Theorem

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In next three to four lectures we will learn how to find the find necessary and sufficient conditions for any topological space to be metrizable.
Definition (Locally Finite Collection)

Let $X$ be topological space. A collection $\mathcal{A}$ is said to be locally finite in $X$ if every point of $X$ has a neighborhood that intersects only finitely many elements of $\mathcal{A}$. 
Local Finiteness II

Example

Let \((\mathbb{R}, R_s)\) is topological space with standard topology, the collection of the interval

\[ A = \{(n, n + 2) : n \in \mathbb{Z}_+\} \]

is locally finite collection in \(\mathbb{R}\).
Local Finiteness III

Example

The collection

$$B = \{(0, 1/n) : n \in \mathbb{Z}_+\}$$

is locally finite in \((0, 1)\) but not in \(\mathbb{R}\).
Local Finiteness IV

Example

The collection

\[ C = \left\{ \left( \frac{1}{n+1}, \frac{1}{n} \right) : n \in \mathbb{Z}_+ \right\} \]

is locally finite in \((0, 1)\) but not in \(\mathbb{R}\).
Definition (Countable Locally Finite Collection)

A collection $\mathcal{B}$ of subsets of $X$ is said to be countably locally finite if $\mathcal{B}$ can be written as the countable union of collection $\mathcal{B}_n$, each of which is locally finite.

Remark

- *Some time use $\sigma$-locally finite for countably locally finite concept.*
- *Both a countable collection and a locally finite collection are countable locally finite.*
Refinement I

Definition (Refinement)

Let \( \mathcal{A} \) be a collection of subsets of \( X \). A collection \( \mathcal{B} \) of subsets of \( X \) is said to be a refinement of \( \mathcal{A} \) (or is said to refine \( \mathcal{A} \)) if for each element \( B \) of \( \mathcal{B} \), there is an element \( A \) of \( \mathcal{A} \) containing \( B \).

If the elements of \( \mathcal{B} \) are open sets, we call \( \mathcal{B} \) an open refinement of \( \mathcal{A} \).
And if they are closed sets, we call \( \mathcal{B} \) is a closed refinement.
Example

Let $\mathcal{A}$ be the following collection of subsets of $\mathbb{R}$:

$$\mathcal{A} = \{(n, n + 2): n \in \mathbb{Z}\}$$

Which of the following collection refine $\mathcal{A}$?

1. $\mathcal{B} = \{(x, x + 1): n \in \mathbb{R}\}$
2. $\mathcal{C} = \{(n, n + \frac{3}{2}): n \in \mathbb{Z}\}$
3. $\mathcal{D} = \{(x, x + \frac{3}{2}): n \in \mathbb{R}\}$
If we have locally finite collection of subsets of $X$ are given, then how can we make other locally finite collection of $X$. Here following properties gives such type of the structure.

**Lemma**

Let $\mathcal{A}$ be a locally finite collection of subsets of $X$, then

1. Any subcollection of $\mathcal{A}$ is locally finite.
2. The collection of closure of the element of $\mathcal{A}$ that is

$$\mathcal{B} = \{\overline{A} \}_{A \in \mathcal{A}}$$

is locally finite.
3. And $\bigcup_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} \overline{A}$. 

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Properties of Locally Finite Collection II

Now we will discuss very important properties of topological space which is metrizable via locally finite collection.

But before this we should know these first:

2. Refinement.
3. Metrizable Space.
4. Open covering.
5. Well-ordering theorem.
Definition (Metrizable Space)

A topological space $X$ is said to be metrizable if there exists a metric $d$ on the set $X$ that induces the topology of $X$. A metric space is a metrizable space $X$ together with a specific metric $d$ that gives the topology of $X$. 
Definition (Open covering)

A collection $\mathcal{A}$ of subsets of a space $X$ is said to cover $X$, or to be a covering of $X$, if the union of the elements of $\mathcal{A}$ is equal to $X$. It is called an open covering of $X$ if its elements are open subsets of $X$. 
Definition (Well-ordered sets)

A set $A$ with an order relation $<$ is said to be well-ordered if every non-empty subset of $A$ has a smallest element.

Theorem (Well-ordering theorem)

If $A$ is a set, there exists an order relation on $A$ that is well ordering.
Lemma

Let $X$ be a metrizable space. If $\mathcal{A}$ is an open covering of $X$, then there is an open covering $\mathcal{E}$ of $X$ that is countable locally finite.

Proof:

Given A topological space $X$ is metrizable.

Given A collection $\mathcal{A}$ of subsets of $X$ is open covering of $X$.

Given A collection $\mathcal{E}$ of subsets of $X$ is also open covering of $X$.

Aim $\mathcal{E}$ is refinement of $\mathcal{A}$ i.e. $\mathcal{E}$ refine to $\mathcal{A}$.

Aim $\mathcal{E}$ is countably locally finite set.
Properties of Locally Finite Collection VII

Choose a well-ordering $<$ for the collection $\mathcal{A}$. Let us denote the elements of $\mathcal{A}$ generically by the letters $U, V, W, \ldots$. Choose a metric for $X$. Let $n$ be fixed positive integer. Now define a subset $S_n(U)$ of $U$ by "shrinking" $U$ a distance of $1/n$. More precisely, let

$$S_n(U) = \{ x : B(x, 1/n) \subset U \} \subset U \subset X$$

Here this set is closed.
Now we use the well-ordering $< \, of \, \mathcal{A}$ to pass to a still smaller set. For each $U$ in $\mathcal{A}$, define

$$T_n(U) = S_n(U) - \bigcup_{V < U} V$$

Now we can see here $\mathcal{A}$ consists of the three sets $U < V < W$ is pictured as:

By figure we can see that sets which we have formed are disjoint.
These sets are separated by a distance of at least $1/n$ i.e. If $V$ and $W$ are distinct elements of $\mathcal{A}$ show that

$$d(x, y) \geq \frac{1}{n}, \text{ whenever } x \in T_n(V) \subset V \text{ and } y \in T_n(W) \subset W$$

Now we will prove this above fact:

Since $x \in T_n(V) \Rightarrow x \in S_n(V)$

$$\Rightarrow \frac{1}{n} - \text{nbd of } x \text{ lies in } V$$  \hspace{1cm} (1)
Properties of Locally Finite Collection $X$

On other hand

Since $V < W, y \in T_n(W)$ and $y \notin V \Rightarrow y \notin \frac{1}{n} - \text{nbd of } x$ \hspace{1cm} (2)

Here $T_n(U)$ is open or not it is not clear. So will make another structure. Let us expand each of sets $T_n(U)$ slightly to obtain an open set $E_n(U)$. Let $E_n(U)$ is $\frac{1}{3n} - \text{nbd of } T_n(U)$. i.e. let $E_n(U)$ be the union of the open balls $B(x, \frac{1}{3n})$ for $x \in T_n(U)$. 
In the case $U < V < W$, we have picture. Form this figure we can see that sets we have formed are disjoint. If $V$ and $W$ are distinct elements of $\mathcal{A}$, we will assert that

$$d(x, y) \geq \frac{1}{3n}, \text{ whenever } x \in E_n(V) \subset V \text{ and } y \in E_n(W) \subset W$$

(And here this fact follows by triangular inequalities.) Note that for each $V \in \mathcal{A}$, the set $E_n(V)$ is contained in $V$. Let us define new structure

$$\mathcal{E}_n = \{ E_n(U) : U \in \mathcal{A} \}$$
Properties of Locally Finite Collection XII

Now we will show that

1. $\mathcal{E}_n$ refinement of $\mathcal{A}$.

2. $\mathcal{E}_n$ is locally finite collection of open sets.

The fact (1) comes from the fact the $E_n(V) \subset V$ for each $V \in \mathcal{A}$. The fact (2) comes from the fact that for any $x \in X$, $\frac{1}{n} - \text{nbd of } x$ can intersect at most one element of $\mathcal{E}_n$. 
Here the collection $\mathcal{E}_n$ will not cover the $X$. So we will assert that the collection

$$\mathcal{E} = \bigcup_{n \in \mathbb{Z}^+} \mathcal{E}_n$$

will cover $X$. 
Let \( x \in X \) be arbitrary and as given \( \mathcal{A} \) cover the \( X \). Let us choose \( U \) be the first element of \( \mathcal{A} \) that contain \( x \).

Since \( U \) is open \( \Rightarrow \) we can choose \( n \) so that \( B(x, \frac{1}{n}) \subset U \)

\[ \Rightarrow x \in S_n(U) \quad (3) \]

Now because \( U \) is the first element that contain \( x \) and the point \( x \in T_n(U) \).

This show that \( x \) will also belong to the element \( E_n(U) \).

Hence we can say that \( \mathcal{E} \) cover \( X \).
Properties of Locally Finite Collection XV

\[ U < V < W \]

\[ T_n(U), T_n(V), T_n(W) \]
Introduction I

In this theorem we will show that regularity of $X$ and existence of a countably locally finite basis for $X$ are equivalent to metrizability. The proof of these condition imply merizability is very closed to the second proof of the Uryshon metrization Theorem.
Introduction II

Here we recall major steps of UMT and comparison with NSMT.

1. To prove that every regular space $X$ with a countable basis is normal.
   \[ \Downarrow \]
   To prove regular space $X$ with a basis that is countably locally finite is normal.

2. To construct a countable collection $\{f_n\}$ of real-valued functions on $X$ that separates points from closed sets.
   \[ \Downarrow \]
   Construct a certain collection of real-valued functions $\{f_\alpha\}$ on $X$ that separates points from closed sets.
To use the functions $f_n$ to define a map imbedding $X$ in the product space $\mathbb{R}^\omega$.

To use the functions $f_\alpha$ to define a map imbedding $X$ in the product space $\mathbb{R}^J$.

To show that if $f_n \leq \frac{1}{n}$ for all $x$, then this map actually imbeds $X$ in the metric space $(\mathbb{R}^\omega, \rho)$.

We show that if the function $f_\alpha$ are sufficiently small, this map actually imbeds $X$ in the metric space $(\mathbb{R}^J, \rho)$.
Before start to proof the theorem we will recall some important facts which will help in proofing the theorem.

Definition ($G_\delta$ set)

A subset of a space is called a $G_\delta$ set in $X$ if it equals the intersection of a countable collection of open subsets of $X$.

Example

1. Each open subset of $X$ is $G_\delta$ set.
2. First-countable Hausdorff space, each one-point set is a $G_\delta$ set.
3. In metric space $X$, each closed set is a $G_\delta$ set.
Lemma

Let $X$ be a regular space with a basis $\mathcal{B}$ that is countable locally finite, then $X$ is normal, and every closed set in $X$ is a $G_\delta$ set in $X$. 
Lemma

Let $X$ be normal and let $A$ be a closed $G_\delta$ set in $X$, then there is a continuous function

$$f : X \rightarrow [0, 1]$$

defined by

$$f(x) = \begin{cases} 
0, & \text{if } x \in A; \\
> 0, & \text{if } x \notin A.
\end{cases}$$
Proof of The Nagata Smirnov Metrization Theorem

Theorem

A space $X$ is metrizable if and only if $X$ is regular and has a basis that is countably locally finite.

Part I

Given A topological space $X$ is metrizable space.
Aim 1 To show that $X$ is regular space.
Aim 2 And $X$ has a countably locally finite basis.

Part II

Given 1 $X$ is regular space.
Given 2 And $X$ has a countably locally finite basis.
Aim $X$ is metrizable space.
Proof of The Nagata Smirnov Metrization Theorem II

Proof of the Part I
As we know that metrizable space is regular so we will proof only IIInd part that is \( X \) has a basis which is countably locally finite.

Choose a metric for \( X \).

Given \( m \), let \( \mathcal{A}_m \) be the covering of \( X \) by all open balls of the radius \( \frac{1}{m} \).

By the lemma, there is an open covering \( \mathcal{B}_m \) of \( X \) refining \( \mathcal{A}_m \) such that \( \mathcal{B}_m \) is countably locally finite.

Note: Each element of \( \mathcal{B}_m \) has diameter at most \( \frac{2}{m} \).

Let \( \mathcal{B} \) be the union of the collection \( \mathcal{B}_m \) for \( m \in \mathbb{Z}_+ \).

Now \( \mathcal{B} \) will be countably locally finite, because each collection \( \mathcal{B}_m \) is countably locally finite.
Now we will show that $\mathcal{B}$ is a basis for $X$.

Given $x \in X$ and given $\epsilon > 0$.

We show that there is an element $B$ of $\mathcal{B}$ containing $x$ that is contained in $B(x, \epsilon)$.

First choose $m$ so that $\frac{1}{m} < \frac{\epsilon}{2}$.

Then, because $\mathcal{B}_m$ covers $X$, we can choose an element $B \in \mathcal{B}_m$ that contains $x$.

Since $B$ contains $x$ and has diameter at most $\frac{2}{m} < \epsilon$, it is contained in $B(x, \epsilon)$. That is required result.
• This the most useful generalization of compactness.
• It is useful in topology and differential geometry.
• Here will discuss only one application, a Metrization Theorem.
• Many of the spaces are familiar to us already are paraparacompact.
How the paracompacteness generalize compactness.
We recall the definition of compactness

Definition (Compact space)

A space $X$ is said to be compact if every open covering $\mathcal{A}$ of $X$ containing a finite subcollection that covers $X$.

Equivalently we can say that

Definition (Compact space)

A space $X$ is said to be compact if every open covering $\mathcal{A}$ of $X$ has a finite open refinement $\mathcal{B}$ that covers $X$. 
Given such refinement $\mathcal{B}$, one can choose for each element of $\mathcal{B}$ an element of $\mathcal{A}$ containing it. And this way one obtain finite collection of $\mathcal{A}$ that covers $X$. This suggest the a way to generalize the compact space:

**Definition (Paracompact space)**

A space $X$ is paracompact if every open covering $\mathcal{A}$ of $X$ has a locally finite open refinement $\mathcal{B}$ that covers $X$. 
Example
The space $\mathbb{R}^n$ is paracompact space.

Let $X = \mathbb{R}^n$ and $\mathcal{A}$ is open covering of $X$. Let $B_m(0, m)$ is the open ball of radius $m \in \mathbb{R}$ and center 0. Here we can obtain $\mathcal{C} = \bigcup C_m$ is refinement of $\mathcal{A}$. Given $m$, choose finitely many element of $\mathcal{A}$ that cover $\overline{B}_m$ and intersect each one with the open set $X - \overline{B}_{m-1}$. This collection is denoted by $\mathcal{C}_m$. The collection $\mathcal{C}$ is locally finite. And finally this collection $\mathcal{C}$ cover $X$. 
Some other example of paracompact spaces are

- Every compact space is paracompact.
- Any second-countable locally compact Hausdorff space is paracompact.
Some properties of a paracompact space are similar of a compact space. Here we shall verify these facts. The subspace of a paracompact space is not necessarily paracompact but

**Theorem**

*Every closed subspace of a paracompact space is paracompact.*

**Proof:**

**Given** Let $Y$ be a closed subspace of paracompact space $X$.

**Aim** We want to show that $Y$ is also paracompact space. i.e. there exists a locally finite refinement $\mathcal{B}$ of $\mathcal{A}$ that cover $Y$. 
where $\mathcal{A}$ also cover $Y$. 
Let $\mathcal{A}$ be a covering of $Y$ by open set in $Y$.
For each $A \in \mathcal{A}$, choose an open set $A' \in X$ such that

$$A' \cap A = Y$$

Now here $\mathcal{A}$ cover $X$ by the open sets $A'$, along with the open set $X - Y$.
Let $\mathcal{B}$ be a locally finite open refinement of this covering $\mathcal{A}$ that covers $X$.
The collection

$$\mathcal{C} = \{B \cap Y : B \in \mathcal{B}\}$$

is the reburied locally finite open refinement $\mathcal{A}$.
Theorem

*Every paracompact Hausdorff space* $X$ *is normal.*

**Proof:**

**Given** Let $X$ be a paracompact Hausdorff space.

**Aim** We want to show that $X$ is normal.

First we prove regularity.

Let $a$ be a point of $X$ and $B$ be a closed set of $X$ disjoint from $a$.

By the Hausdorff condition we can choose for each $b \in B$, an open set $U_b$ about whose closure is disjoint from $a$.

Cover $X$ by the open sets $U_b$, along with open set $X - B$, consider a locally finite refinement $C$ that covers $X$.

From the subcollection $D$ of $C$ consisting of every element of $C$ that intersect $B$. Then this show that $D$ cover $B$.

And if $D \in D$, then $\overline{D}$ will disjoint from $a$. 
For $D$ intersect $B$, so it lies in some set $U_b$, whose closure is disjoint from $a$.

Let

$$V = \bigcup_{D \in \mathcal{D}} D$$

Then $V$ is an open set in $X$ containing $B$.

Because $\mathcal{D}$ is locally finite

$$\overline{V} = \bigcup_{D \in \mathcal{D}} \overline{D}$$

so that $\overline{V}$ is disjoint from $a$.

By the definition of regularity we can say that $X$ is regular.
Same way we can show that $X$ is normal.

- Replace $a$ by the closed set $A$.
- Replace Hausdorff condition by regularity.
As like that every metrizable space is compact we can also study the metrizability for paracomapct space. But before this following lemma is very useful.

**Lemma**

Let $X$ be regular. Then the following condition on $X$ are equivalent. Every open covering of $X$ has a refinement that is

1. An open covering of $X$ and locally finite.
2. A covering of $X$ and locally finite.
3. A closed covering of $X$ and locally finite.
4. An open covering of $X$ and locally finite.
Theorem (Metrizability of Paracompact Space)

Every metrizable space is paracompact.

Proof:
Let $X$ be a metrizable space. Then by the lemma, given an open covering $\mathcal{A}$ of $X$, it has an open refinement that covers $X$ and is countably locally finite. The last lemma show that $\mathcal{A}$ has an open refinement that covers $X$ and is locally finite.
Definition (Lindelof Space)

A topological space $X$ is called Lindelof space if every open cover of $X$ is reduciable to a countable cover.

Remark

On comparing the definition of compact and Lindelof space we can say that

Every compact space is Lindelof space but its converse is not true.

For example $(X, D)$ is Lindelof space where $X$ is any infinite countable set but it is not compact.
Theorem

Every regular Lindelof space is paracompact.

Proof:
Let $X$ be regular Lindelof space.

Given an open covering $\mathcal{A}$ of $X$, it has a countable subcollection that cover $X$. And this subcollection will also countably locally finite.

By the lemma, we can show that $\mathcal{A}$ has an open refinement that covers $X$ and is locally finite.
Example (Product of Paracompact Spaces)

The product of two paracompact spaces need not be paracompact. Here we can see that $\mathbb{R}_l$ is paracompact because it is regular Lindelof space. But $\mathbb{R}_l \times \mathbb{R}_l$ is not paracompact because it is Hausdorff but not normal.
Example (Paracompactness of \( \mathbb{R}^\omega \))

The space \( \mathbb{R}^\omega \) is paracompact in both product and uniform topologies. This is due to \( \mathbb{R}^\omega \) is metrizable in both topologies.
Example (Paracompactness of $\mathbb{R}^J$)

The product space $\mathbb{R}^J$ is not paracompact if $J$ is uncountable. Because $\mathbb{R}^J$ is Hausdorff but not Normal.
The Smirnov metrization theorem gives another set of necessary and sufficient condition for a metrizability of a space. It is a corollary of Nagata-Smirnov metrization theorem. But before discussing this here we will introduce new form of metrizability as locally metrizability.
Definition (Locally metrizable space)

A space $X$ is locally metrizable if every point $x$ of $X$ has a neighborhood $U$ that is metrizable in the subspace topology.

In other way we can say that a topological space is termed locally metrizable if it satisfies the following equivalent conditions:

1. Every point has an open neighbourhood which is a metrizable space.
2. Given a point, and any open neighbourhood of the point, there is a smaller open neighbourhood of the point which is metrizable.

Smirnov proved that a locally metrizable space is metrizable if and only if it is Hausdorff and paracompact.
Theorem (Smirnov Metrization Theorem)

A space $X$ is metrizable if and only if it is a paracompact Hausdorff space that is locally metrizable.

Proof: Part-I

**Given** Let $X$ is metrizable space.

**Aim** We want to show that $X$ is paracompact Hausdorff space.

**Aim** And $X$ is locally metrizable.

As metrizability is strong form compare to locally metrizability so if $X$ is metrizable then it will also locally metrizable space.

By the result $X$ will be paracompact also.
Proof: Part-II

Given  Let $X$ is paracompact Hausdorff space space.

Given  And $X$ is locally metrizable.

Aim  We want to show that $X$ is metrizable space.
     i.e. We shall show that $X$ has a basis that is countably locally finite.

Since $X$ is regular, then it will follow the NST, so $X$ will metrizable space.
Claim I:
There exists a basis $\mathcal{B}$ for the topology of $X$, of type $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$, where each $\mathcal{B}_n$ is a locally finite family. Moreover, for each $B \in \mathcal{B}$, there is a continuous function

$$f_B : X \to [0, 1]$$
such that $B = \{x \in X : f_B(x) \neq 0\}$

Proof:
By hypothesis, there is an open cover $\mathcal{U} = \{U_i : i \in I\}$ of $X$, on which the topology is induced by a metric $d_i$. Let us assume that $d_i \leq 1$.

For each $i \in I$, we denote by $B_i(x, r)$ the balls induced by $d_i$. They are open subsets of $U_i$, hence also open in $X$.

By the shrinking lemma, we can find another locally finite cover $\{V_i : i \in I\}$ with $\overline{V_i} \subset U_i$.

For each integer $n$, we consider the open cover of $X$

$$\{B_i(x, \frac{1}{n}) \cap V_i : i \in I, x \in U_i\}$$
Let $B_n$ be a locally finite refinement of it and $B = \bigcup_n B_n$. For each $B \in B$, we find $i$ such that $B \in V_i$ and then $f_B(x) := d_i(x, U_i B)$ is a well-defined continuous function on $U_i$ with which is zero outside $B$; since $\bar{B} \subset \bar{V}_i \subset U_i$ (where all the closures are in $X$), extending $f_B$ by zero outside $U_i$, it will give us a function with the desired properties.

Finally, we show that $B$ is a basis. Consider $U \subset X$ open, $x \in U$; we show that $x \in B \subset U$ for some $B \in B$. Since $U$ is locally finite, there is only a finite set of indices $i$ with $x \in U_i$; call it $F_x$. For each $i \in F_x$, $U \cap U_i$ is open in $(U_i, d_i)$ hence we find $\epsilon_i$ such that $B_i(x, \epsilon_i) \subset U \cap U_i$. Choose $m$ with $2/m < \epsilon_i$ for all $i \in F_x$. Choose $B \in B_m$ such that $x \in B$; due to the definition of $B_m$, we have $B \subset B_i(y, 1/m)$ for some $i \in I, y \in U_i$. In particular, $x \in U_i$, hence $i \in F_x$. From the choice of $m$, we have $B_i(y, 1/m) \subset B_i(x, \epsilon_i)$; from the choice of $\epsilon_i$, these are inside $U$. 
Claim II:
The following is a metric on $X$ inducing the topology $\tau$ on $X$.

$$d: X \times X \rightarrow \mathbb{R}$$

defined by

$$d(x, y) = \sup\{\frac{1}{n}|f_B(x) - f_B(y)| : n \geq 1 \text{ integer, } B \in \mathcal{B}_n\}$$

Proof:
By the same argument as in the UMT, $d$ is a metric.
Here we will show that $\tau = \tau_d$ and for this we will show that $\tau \subset \tau_d$ and $\tau_d \subset \tau$.
Let $U \subset X$ open, $x \in U$. We have to find $r > 0$ such that $B_d(x, r) \subset U$.
Since $\mathcal{B}$ is a basis, we find $B \in \mathcal{B}_n$ for some $n$, with $x \in B \subset U$. We claim that $r = \frac{1}{n}|f_B(x)|$ does the job. Indeed, if $y \in B_d(x, r)$, we have

$$\frac{1}{n}|f_B(y)f_B(x)| < \frac{1}{n}|f_B(x)|,$$

hence $f_B(y) \neq 0$, hence $y \in B$, hence $y \in U$. 
Finally, we show that $\tau_d \subset \tau$. It suffices to show that, for any $x \in X, r > 0$, there exists $U \in \tau$ such that $x \in U \subset B(x, r)$. Let $n_0 > 2/r$ be an integer. Since each $B_n$ is locally finite, we find a neighborhood $V$ of $x$ which intersects only a finite number of $B$s with $B \in B_n, n \leq n_0$. Call these members $B_1, \ldots, B_k$. Choose $U \subset V$ such that

$$|f_{b_i}(y) - f_{b_i}(x)| < r, \forall y \in U, \forall i \in \{1, \ldots, k\}$$

We claim that $U \subset B(x, r)$. That means that, for any $y \in U$, we have $\frac{1}{n}|f_B(y) - f_B(x)| < r$ for all $n \geq 1$ and $B \in B_n$. If $n \geq n_0$ this is automatically satisfied since $|f_B| \leq 1$ and $2/n \leq 2/n_0 < r$. Assume now that $n \leq n_0$. If $B$ is not one of the $B_1, \ldots, B_k$, then $U \cap B = \phi$ hence $f_B(y) = f_B(x) = 0$ and we are done. Finally, if $B = B_i$ for some $i$, then the desired inequality follows from (4).
Definition (Regular Space)

Suppose that one-point sets are closed in $X$. Then $X$ is said to be regular if for each pari consisting of a point $x$ and a closed set $B$ disjoint from $x$, there exist disjoint open sets containing $x$ and $B$, respectively.
Definition (Normal Space)

The space $X$ is said to be normal if for each pair $A, B$ of disjoint closed sets of $X$, there exist disjoint open sets containing $A$ and $B$, respectively.
Definition (Housdroff Space)

A space $X$ is called a Housdroff space if for each pair $x_1, x_2$ of distinct points of $X$, there exist neighborhood $U_1$ and $U_2$ of $x_1$ and $x_2$, respectively, that are disjoint.
Lemma (Shrinking Lemma)

If $X$ is a paracompact Hausdorff space then $X$ is normal and, for any open cover $\mathcal{U} = \{U_i : i \in I\}$ there exists a locally finite open cover $\mathcal{V} = \{V_i : i \in I\}$ with the property that $\overline{V_i} \subset U_i$ for all $i \in I$. 
References

Lecture Notes on Paracomactness and Locally Compactness
Lecture Notes on Urysohn and Smirnov Metrization Theorem
Lecture Notes on an exploration of metrizibility of topological spaces
Metrizable Topological Space Notes

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Feel free to mail me for any type of suggestion for improvement of this lecture notes: drsanjaymishra1@gmail.com
Thank you!