Homotopy of Path

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The basic problem of topology is to determine whether two given topological spaces are homeomorphic or not. Generally we have not any any techniques by which we can solve this problem but particulary we can solve this for some spaces.
There are two questions arises for homeomorphism

1. How to show that two spaces are homeomorphic?
2. How to show that two spaces are not homeomorphic?

The solution are

1. Solution of first question is construction of continuous functions on one space to other space such way that it inverse continues function also exists. With the help of some techniques we can construct such function.
But solution of second question is totally different to first solution. For this we will find a continuous function which of inverse continuous not exits. Or, we can solve this problem in another way also as like if one can find some topological property that holds for one space but not for other, then the problem is solved and we can say that spaces are not homeomorphic.
Example

The closed interval $[0, 1]$ and open interval $(0, 1)$ are not homeomorphic because first space is compact but second is not.
Example

The real line $\mathbb{R}$ and "long line" $L$ are not homeomorphic because first space has a countable basis but second is not or we can say that first space is second countable but second space is not.
Example

The real line $\mathbb{R}$ and and plane $\mathbb{R}^2$ are not homeomorphic because when deleting a point from $\mathbb{R}^2$ leaves a connected space remaining but deleting a point from $\mathbb{R}$ does not.
All last three examples, we can see that topological properties help to distinguishes the spaces. But in general it is not helpful for example if we consider the plane $\mathbb{R}^2$ and three-dimensional $\mathbb{R}^3$, here topological properties as like compactness, connectedness, local connectedness, metrizability and so on are not distinguishes these two spaces.
Consider another very interesting example such surfaces as 2-sphere $S^2$ and the torus $T$ (surface of doughnut) and the double torus $T\#T$ (surface of a two-holed doughnut).
None of the topological properties we have studied up to now will distinguish between them. In this situation a new topological properties as simple connectedness help to distinguishes the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ and we can say that both are not homeomorphic because delating a point from $\mathbb{R}^3$ leaves a simply connected space remaining but deleting a point from $\mathbb{R}^2$ does not.
It will also distinguishes between 2-sphere $S^2$ and torus $T$ where first is simply confected space but second is not so we can say that both are not homeomorphic.
But we have not any solution on homeomorphism between two spaces $T$ and $T\#T$ through the simply connectedness properties because both are not simply connected.
Now we have more general idea compare to simply connectedness which help to solve the problems of homeomorphism of spaces and that is the **fundamental group** of spaces. This idea includes the simple connectedness as special case. Now we can say that two spaces are homeomorphic if the fundamental groups of both spaces are **isomorphic**. And the condition of simple connectedness is just the condition that the fundamental group of space $X$ is the trivial (one-element) group. For example, if we want to show that $S^2$ and $T$ are not homeomorphic we will just show that $S^2$ have trivial fundamental group but $T$ have not. Similarly $T$ and $T\#T$ are not homeomorphic because $T$ has an abelian fundamental group but $T\#T$ has has not.
Homotopy of Paths I

Homotopy theory, which is the main part of algebraic topology, studies topological objects up to homotopy equivalence. Homotopy equivalence is a weaker relation than topological equivalence, i.e., homotopy classes of spaces are larger than homeomorphism classes. Even though the ultimate goal of topology is to classify various classes of topological spaces up to a homeomorphism, in algebraic topology, homotopy equivalence plays a more important role than homeomorphism, essentially because the basic tools of algebraic topology (homology and homotopy groups) are invariant with respect to homotopy equivalence, and do not distinguish topologically nonequivalent, but homotopic objects.
Homotopy of Paths II

Definition (Homotopy)

Suppose there are two continuous maps \( f, g: X \to Y \) between topological spaces \( X \) and \( Y \), then we call \( f \) is homotopic to \( g \), there exists a continuous map \( F: X \times I \to Y \) such that

\[
F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = g(x), \quad \forall x \in X
\]

Here \( I = [0, 1] \) is unit interval and the map \( F \) is called a homotopy between \( f \) and \( g \). And if \( f \) is homotopic to \( g \) then we write \( f \simeq g \).
More intuitively, if we think of the second parameter of $F$ as time, then $F$ describes a continuous deformation of $f$ into $g$. At time $t = 0$ we have the function $f$, at time $t = 1$ we have the function $g$. In other words, homotopy is continuous one-parameter family of maps from $X$ to $Y$ and if imagine that parameter $t$ is as time then the homotopy $F$ represent a continuous "deformation" of map $f$ to $g$ as $t$ goes from 0 to 1.
Definition (Nullhomotopy)

If a map $f : X \rightarrow Y$ is called null-homotopic it $f$ is homotopic to constant map $c : X \rightarrow \{y_0\} \subset Y$.
Example

The function $f : \mathbb{R} \to \mathbb{R}$ is identity map defined by $f(x) = x$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x + 1$, then a map $F : \mathbb{R} \times I \to \mathbb{R}$ defined by $F(x, t) = x + t$ is the homotopy between $f$ and $g$. Here we can see that for a fixed value of $t$, the homotopy $F(x, t)$ translate the real line $\mathbb{R}$ a distance $t$. As $t$ varies from 0 to 1, $F$ takes us from $f$ to $g$. 
Example

Compressing the real line interval \([-100, 100] \in \mathbb{R}\) into a point \(\{0\}\). The required homotopy would be \(F(x, t) = x(1 - t)\).
The identity map $id: \mathbb{D}^2 \to \mathbb{D}^2$ and the constant map $c_0: \mathbb{D}^2 \to 0 \in \mathbb{D}^2$ of the disk $\mathbb{D}^2$ are homotopic. A homotopy between them defined by $F(t, (\rho, \varphi)) = ((1 - t) \cdot \rho, \varphi)$, where $(\rho, \varphi)$ are polar coordinates in $\mathbb{D}^2$. Thus the identity map of the disk is null homotopic.
Now if we consider functions as paths in topological space $X$, then homotopy between them is stronger than defined earlier.
Homotopy of Paths IX

Definition (Path Homotopy)

Two paths $f : I \to X$ and $g : I \to X$ in $X$ are said to be path homotopy if they have same initial point as $f(0) = g(0) = x_0$ and the same final point as $f(1) = g(1) = x_1$ and if there is a continuous function $F : I \times I \to X$ such that

$$F(s, 0) = f(s) \text{ and } F(s, 1) = g(s)$$

$$F(0, t) = x_0 \text{ and } F(1, t) = x_1$$

foe each $s \in I$ and each $t \in I$. And we call $F$ is path homotopy between $f$ and $g$ and write it as $f \simeq_p g$. 
Homotopy of Paths $X$
Example

The two continuous functions as paths on plane is given as $f : I \to \mathbb{R}^2$ defined by $f(x) = (\cos(\pi x), -\sin(\pi x))$ and $g : I \to \mathbb{R}^2$ defined by $g(x) = (\cos(\pi x), \sin(\pi x))$. Here path $f$ is homotopic to $g$ and homotopy $F : I \times I \to \mathbb{R}^2$ is defined by $F(x, t) = (\cos(\pi x), (1 - 2t)\sin(\pi x))$. 
Theorem

Homotopy is an equivalence relation.

Proof:
We need to show that a homotopy is

- Reflexive  It is trivial to show that $f \simeq f$. The required map is $F(x, t) = f(x)$.

- Symmetric Let $f \simeq g$, thus there exist a map $F: X \times I \to Y$ such that

\[
F(x, 0) = f(x) \\
F(x, 1) = g(x) \quad \forall x \in X
\]
Define $G: X \times I \to Y$ such that

\[ G(x, t) = F(x, 1 - t) \]

Therefore,

\[ G(x, 0) = g(x) \]
\[ G(x, 1) = f(x) \]

which implies that $g \simeq f$. 
Transitive  Let $f \simeq_F g$ and $f \simeq_G g$ be maps from $X$ to $Y$. Define

$$H(x, t) = \begin{cases} 
F(x, 2t), & t \in [0, \frac{1}{2}], \\
G(x, 2t - 1), & t \in [\frac{1}{2}, 1].
\end{cases}$$

By simple substitution, we get the following results:

$$H(x, 0) = F(x, 2t) = f(x)$$
$$H(x, \frac{1}{2}) = F(x, 1) = G(x, 0) = g(x)$$
$$H(x, 1) = G(x, 1) = h(x)$$

Since the deformations are continuous within the two intervals of $t$, we can conclude that $f$ is deformed into $g$ from $t \in [0, \frac{1}{2}]$ and then deformed into $h$ from $t \in [\frac{1}{2}, 1]$. Therefore $f \simeq_H h$. 
Homotopy of Paths XV

Theorem

Path homotopy is an equivalence relation.
Definition (Homotopy Equivalence)

Given two spaces $X$ and $Y$ are said to be homotopy equivalent or same homotopy type if there exists continuous maps $f: X \to Y$ and $g: X \to Y$ such that

$$gof = id_x \text{ and } fog = id_y$$
Homotopy Equivalence II

Remark

1. *Every homeomorphism is a homotopy equivalence but its converse is not true.*
   For example a solid disk is not homeomorphic to a single point since there is no bijection between them although the disk and the point are homotopic equivalent since we can deform the disk along the radial lines continuously to a single point.

2. *Space that are homotopy equivalent to a point are called contractible.*
Before discussing the identities, inverse and associative properties of homotopy equivalence class we will first discuss two important facts:

1. If \( k: X \to Y \) is a continuous map and if \( F \) is a path homotopy in \( X \) between the path \( f \) and \( g \), then \( k0F \) is a path homotopy in \( Y \) between the paths \( kof \) and \( kog \).

2. If \( k: X \to Y \) is a continuous map and if \( f \) and \( g \) are paths in \( X \) with \( f(1) = g(0) \), then

\[
k0(f \ast g) = (kof) \ast (kog)
\]
Homotopy Equivalence IV

Theorem (Left and Right Identities)

Given $x \in X$ let $e_x$ denote the constant path $e_x : I \to X$ carrying all of $I$ to the paths $x$. If $f$ is a path in $X$ from $x_0$ to $x_1$, then

$$ [f] * [e_{x_1}] = [f] \text{ and } [e_{x_0}] * [f] = [f] $$

Proof:

Let $e_0 : I \to X$ is constant path in $I$ at 0 and let $i : I \to I$ is identity map (if is also a path in $I$ form 0 to 1). Then $e_0 * i : I \to X$ is path in $I$ from 0 to 1.

Because $I$ is convex set, there is a path homotopy $G$ in $I$ between $i$ and $e_0 * i$. 
Definition

Let $\alpha: I \rightarrow X$ be a path in $X$, then a map

$\hat{\alpha}: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ defined by

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$$

This map $\hat{\alpha}$ is well defined because $*$ is well defined. If $f$ is a loop at $x_0$ then $\bar{\alpha} * (f * \alpha)$ is also a loop at $x_1$.

Here $\hat{\alpha}$ depends only on the path-homotopy class of $\alpha$. 
Theorem

The map $\hat{\alpha} : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ defined by

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$$

is a group isomorphism.

Proof:

First we show that $\hat{\alpha}$ is a homomorphism. We compute

$$\hat{\alpha}([f]) * \hat{\alpha}([g]) = ([\bar{\alpha}] * [f] * [\alpha]) * ([\bar{\alpha}] * [g] * [\alpha]) \quad \text{(By definition of) } \hat{\alpha}$$

$$= ([\bar{\alpha}] * [f]) * ([\alpha] * [\bar{\alpha}]) * ([g] * [\alpha])$$

$$= [\bar{\alpha}] * [f] * [g] * [\alpha]$$

$$= \hat{\alpha}([f] * [g])$$
Now to show that \( \hat{\alpha} \) is an isomorphism, we show that if \( \beta \) denotes the path \( \bar{\alpha} \), which is reverse of \( \alpha \), then \( \hat{\beta} \) is an inverse of \( \hat{\alpha} \). Now compute for each element \([h]\) of \( \pi_1(X, x_1)\).

\[
\hat{\beta}([h]) = [\bar{\beta}] * [h] * [\beta] \\
= [\alpha] * [h] * [\bar{\alpha}]
\]

And

\[
\hat{\alpha}(\hat{\beta}([h])) = [\bar{\alpha}] * ([\alpha] * [h] * [\bar{\alpha}]) * [\alpha] \\
= ([\bar{\alpha}] * [\alpha]) * [h] * ([\bar{\alpha}] * [\alpha]) \\
\hat{\alpha}(\hat{\beta}([h])) = [h], \ \forall \ [h] \in \pi_1(X, x_1)
\]
And similarly we can show that

\[ \hat{\beta}(\hat{\alpha}([f])) = [f], \quad \forall [f] \in \pi_1(X, x_0) \]

Hence by above two result we can say that \( \hat{\alpha} \) is isomorphism.
Corollary

If $X$ is path connected and $x_0$ and $x_1$ are two points in $X$, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$. 
Fundamental Group VI

How the path-connected space is important for the study of fundamental group?

If $X$ is space and let $C$ be the path-component of $X$ contain $x_0$, then we can show that

$$\pi_1(C, x_0) = \pi_1(X, x_0)$$

And this is because all loops and homotopies in $X$ are based at $x_0$ must lie in the subspace $C$ of $X$.

Remark

If $X$ is path-connected, then all the groups $\pi_1(X, x)$ are isomorphic.
Definition (Simply Connected Space)

A space $X$ is said to be simply-connected if it is a path-connected space and if $\pi_1(X, x_0)$ is trial group for some $x_0 \in X$, and hence for every $x_0 \in X$. 
Lemma

In a simply-connected space $X$, any two paths having the same initial and final points are path homotopic.

Proof:
Let $\alpha : I \to X$ and $\beta : I \to X$ ate two paths from $x_0$ to $x_1$ in $X$, then $\alpha \ast \overline{\beta}$ will also defined and it will be a loop on $X$ based at $x_0$. Since $X$ is simply-connected space, then loop $\alpha \ast \overline{\beta}$ is path homotopic to the contact loop at $x_0$, then

$$[\alpha \ast \overline{\beta}] = [e_{x_0}] \ast [\beta]$$

From which it follows that $[\alpha] = [\beta]$
Is it true that fundamental group is a topological invariant of the space $X$ or not?
Before investigation of this fact that we will introduce the new notation "Homomorphism induced by a continuous map."
Let \( h : X \to Y \) is a continuous map that carries the point \( x_0 \) to the point \( y_0 \) of \( Y \). We can denote this fact as like

\[
h : (X, x_0) \to (Y, y_0)
\]

If \( f \) is any loop in \( X \) based at \( x_0 \), the composite map \( hof : I \to Y \) will be a loop in \( Y \) based at \( y_0 \).

The correspondence \( f \to hof \) gives to a map carrying \( \pi_1(X, x_0) \) into \( \pi_1(Y, y_0) \).
Definition (Homomorphism induced by continues map)

Let \( h : (X, x_0) \to (Y, y_0) \) be a continuous map, then a map

\[ h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0) \]

defined by

\[ h_*([f]) = [hof] \]

is called the homomorphism induced by \( h \) relative to the base point \( x_0 \).
Fundamental Group XII

Remark

1. The map \( h_* \) is well defined.
2. The map \( h_* \) is homomorphism.
3. The map \( h_* \) depends not only on the map \( h \) but also the choice of the base point \( x_0 \).
Fundamental Group XIII

Theorem

If $h: (X, x_0) \to (Y, y_0)$ and $k: (Y, y_0) \to (Z, z_0)$ are continuous map, then

$$(koh)_*([f]) = k_*o h_*$$

and if $i: (X, x_0) \to (X, x_0)$ is identity map, then $i_*$ is the identity homomorphism.

Proof:
By the definition,

$$(koh)_*([f]) = [(koh)of]$$
(k_*oh_*)((f)) = k_*(h_*[f])
= k_*([hof])
= [k(o(h)f)]
= [(koh)of]

Hence by above we can say that

\[(koh)_* = k_*oh_*\]

Similarly, we can also prove that

\[i_*[f] = [iof] = [f]\]
Corollary

If \( h: (X, x_0) \to (Y, y_0) \) is homomorphism of \( X \) with \( Y \), then \( h_* \) is an isomorphism of \( \pi_1(X, x_0) \) with \( \pi_1(Y, y_0) \).

Proof:

As given \( h: (X, x_0) \to (Y, y_0) \) is homomorphism, let \( k: (Y, y_0) \to (X, x_0) \) be inverse of \( h \), then

\[
k_*oh_* = (koh)_* = i_*
\]

(Because \( i \) is identity map in \( (X, x_0) \))

and

\[
h_*ok_* = (hok)_* = j_*
\]

(Because \( j \) is identity map in \( (Y, y_0) \))
Fundamental Group XVI

Since $i_*$ and $j_*$ are identity homomorphism of the group $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ respectively and $k_*$ is the inverse of $h_*$. Hence with the properties of isomorphism we can say that $h_*$ is isomorphism.
Theorem

Let \( X \) and \( Y \) be path-connected topological spaces, then

\[
\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)
\]

Proof:
Recall that the product topology has the property that a map \( f : Z \to X \) is continuous iff its projections to each factor are continuous. This means a loop based at \((x_0, y_0)\) is equivalent to a pair of loops in \( X \) and \( Y \) at \( x_0 \) and \( y_0 \). Likewise, a homotopy of loops at \((x_0, y_0)\) is equivalent to a pair of homotopies of loops at \( x_0 \) and \( y_0 \). Thus we have a bijection

\[
\pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)
\]

sending \([f] \to ([p_1(f)], [p_2(f)])\). It's obviously a group homomorphism, and hence isomorphism.
Lemma (Pasting Lemma)

Let $X = A \cup B$, where $A$ and $B$ are closed sets in $X$ and let $f : A \rightarrow Y$ and $g : B \rightarrow Y$ are continuous functions. If $f(x) = g(x)$ for all $x$ in $A \cap B$, then there exists a continuous function $h : X \rightarrow Y$ such that

$$h(x) = \begin{cases} f(x), & x \in A; \\ g(x), & x \in B. \end{cases}$$
Appendix II

Definition (Path-component)

A relation on space by defining \( x \sim y \) if there is a path in \( X \) from \( x \) to \( y \). The equivalence classes are called the path-component of \( X \).